Particle acceleration and entropy considerations*

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Abstract

Possible entropy constraints on particle acceleration spectra are discussed.

Solar flare models invoke a variety of initial distributions of the primary energy release over the particles of the flare plasma – ie., the partition of the energy between thermal and nonthermal components. It is suggested that, while this partition can take any value as far as energy is concerned, the entropy of a particle distribution may provide a useful measure of the likelihood of its being produced for a prescribed total energy.

The Gibbs' entropy is calculated for several nonthermal isotropic distribution functions f, for a single particle species, and compared with that of a Maxwellian, all distributions having the same total number and energy of particles. Speculations are made on the relevance of some of the results to the cosmic ray power-law spectrum, on their relation to the observed frequency distribution of nonthermal flare hard X-ray spectrum parameters and on the additional energy release required to achieve lower entropy fs.

Keywords: entropy; acceleration; flares; cosmic rays.

1 Introduction

Models in which energetic particles are invoked as a major component of primary energy release in the impulsive phase of solar flares have been both popular and controversial for decades (see eg, reviews (Brown, 1991; Simnett, 1991)). Despite ever improving data and theoretical modelling, however, it remains unresolved how important such particles are in flare energy transport. Brown and Smith (1980) suggested that the lower entropy of accelerated particles compared to a pure Maxwellian distribution might place a theoretical thermodynamic constraint on the fraction of magnetic energy release likely to go into such acceleration. Brown (1993) has raised this issue again and illustrated the point with a simple example of the thermal/nonthermal entropy difference.

In this paper we put this question on a more quantitative basis by examining the entropy of more general particle distribution functions. Essentially the idea is that, solely on energy conservation grounds (first law of thermodynamics), deposition of energy E among N cold particles could equally well result in (for example) a Maxwellian of temperature $2E/3Nk$, a single particle of energy E with the rest cold, or a delta function distribution of all the particles at particle energy E/N. In a statistical sense, however, it is obvious that the last two are much less probable than the first – ie., they have much lower entropy. (Evolution of a plasma distribution function f from a cold Maxwellian to a nonthermal form with high mean energy does of course involve a particle entropy increase and so is not precluded by the second law of thermodynamics. However, while not *impossible* such an outcome is statistically *much less probable* than production of a hot Maxwellian f since it has lower entropy). Quantifying how much lower the entropy (probability) is should help constrain, in a statistical sense, the likely forms of particle distributions produced by prescribed energy release conditions or conversely constrain the energy release conditions required to yield a prescribed particle distribution.

We recognise at the outset that the present calculations are exploratory and rest on simplifying assumptions which need further study. In particular we assume that entropy considerations are relevant – this raises issues regarding equilibrium and closure of the system, for example, which have been addressed in broad terms by Tolman (1959, pp. 560-564) – but we feel that this will be so provided we deal only with newly-accelerated particles *outside* the (presumably small) accelerating volume. Secondly, we compare here only the entropies of alternative forms of the *particle* distribution function which could be generated by deposition of energy E among N particles in a pre-

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scribed volume V. We neglect, for the moment, entropy contributions from other system degrees of freedom and, in particular, from plasma waves, and from the changing magnetic field presumed responsible for generating the flare particle distribution. The possible importance of including plasma waves as additional degrees of freedom in considering the evaluation of plasma entropy during particle acceleration has been discussed in (Grognard, 1983). Our neglect of magnetic field entropy is based on the heuristic argument that the field is associated solely with an ordered zero entropy flow component of the particles – the Gibbs entropy $\sim \iiint f \ln f \, \mathrm{d}^3\mathbf{v}$ of any particle distribution function $f(\mathbf{v})$ is unchanged by adding a systematic drift $\mathbf{v}_0$ (ie., $\mathbf{v} \to \mathbf{v} + \mathbf{v}_0$).

2 Definitions

In this paper we will deal solely with the Gibbs entropy S of the 3-D (velocity space) distribution function $f(\mathbf{r}, \mathbf{v})$ for a single species of nonrelativistic particle of mass m, defined by

$$S = -k \int_{\mathbf{v}} \int_{\mathbf{r}} f \ln f \, \mathrm{d}^3\mathbf{r} \, \mathrm{d}^3\mathbf{v} \tag{1}$$

where k is Boltzmann's constant. For simplicity, we consider only distributions in which the velocity $\mathbf{v}$ distribution is separable from the configuration space $\mathbf{r}$ dependence. With this restriction, we may write

$$f(\mathbf{r}, \mathbf{v}) = n(\mathbf{r}) G(\mathbf{v}) \tag{2}$$

where $n(\mathbf{r})$ is the total particle space density at $\mathbf{r}$ and

$$\int_{\mathbf{v}} G(\mathbf{v}) \, \mathrm{d}^3\mathbf{v} = 1 \tag{3}$$

Because (1) is not linear in f, the absolute value of S depends on the velocity and coordinate units used. Here, however, we will only be concerned with entropy differences between different $G(\mathbf{v})$. If we take some characteristic speed v_0 as velocity unit and write $\mathbf{v} = v_0 \mathbf{u}$ and $G(\mathbf{v}) \to g(\mathbf{u})/v_0^3$ then (1) and (3) yield

$$\begin{aligned} S &= -k \int_{\mathbf{u}} \int_{\mathbf{r}} n(\mathbf{r}) g(\mathbf{u}) \ln[n(\mathbf{r}) g(\mathbf{u})/v_0^3] \, \mathrm{d}^3\mathbf{r} \mathrm{d}^3\mathbf{u} \\ &= -k \int_{\mathbf{r}} n(\mathbf{r}) \ln n(\mathbf{r}) \, \mathrm{d}^3\mathbf{r} \int_{\mathbf{u}} g(\mathbf{u}) \, \mathrm{d}^3\mathbf{u} \\ &\quad - k \int_{\mathbf{r}} n(\mathbf{r}) \, \mathrm{d}^3\mathbf{r} \int_{\mathbf{u}} g(\mathbf{u}) \ln g(\mathbf{u}) \, \mathrm{d}^3\mathbf{u} \\ &\quad + k \ln v_0^3 \int_{\mathbf{r}} n(\mathbf{r}) \, \mathrm{d}^3 r \int_{\mathbf{u}} g(\mathbf{u}) \, \mathrm{d}^3\mathbf{u} \\ &= -kN_0 \int_{\mathbf{u}} g(\mathbf{u}) \ln g(\mathbf{u}) \, \mathrm{d}^3\mathbf{u} \\ &\quad + kN_0 \left\{ \ln v_0^3 - \frac{1}{N_0} \int_{\mathbf{r}} n(\mathbf{r}) \ln n(\mathbf{r}) \, \mathrm{d}^3\mathbf{r} \right\} \end{aligned} \tag{4}$$

where

$$N_0 = \int_{\mathbf{r}} n(\mathbf{r}) \, \mathrm{d}^3\mathbf{r} \tag{5}$$

is the total number of particles. If we consider different distribution functions f with the same *spatial* distribution $n(\mathbf{r})$ and total volume (ie., same N_0) and use the same velocity unit v_0 throughout then we can measure the *entropy differences* by the 'scaled entropy'

$$\Sigma = \frac{1}{kN_0} \left(S - kN_0 \left\{ \ln v_0^3 - \frac{1}{N_0} \int_{\mathbf{r}} n(\mathbf{r}) \ln n(\mathbf{r}) \, \mathrm{d}^3\mathbf{r} \right\} \right) \tag{6}$$

so that by (5), Σ can be expressed solely in terms of $g(\mathbf{u})$, via

$$\Sigma\{g\} = - \int_{\mathbf{u}} g(\mathbf{u}) \ln g(\mathbf{u}) \, \mathrm{d}^3\mathbf{u} \tag{7}$$

and we note, in terms of g, normalisation (3) is

$$\int_{\mathbf{u}} g(\mathbf{u}) \, \mathrm{d}^3\mathbf{u} = 1. \tag{8}$$

We do not consider further here the spatial structure contribution to the entropy (last term in (6)) – i.e., we only consider entropies of different $g(u)$ but the same $n(\mathbf{r})$.Also, throughout the rest of this paper we consider only isotropic $f(\mathbf{v})$ so that $\int_{\mathbf{u}} \mathrm{d}^3\mathbf{u} \to \int_0^\infty 4\pi u^2 \, \mathrm{d}u$. We will be comparing the relative entropies of various distributions g with that of a pure Maxwellian of the same N_0 and of the same total energy E (ie., we are comparing identical plasmas with the same total energy deposited in them). It is therefore convenient to use as velocity unit v_0, the thermal speed in the Maxwellian plasma

$$v_0 = (2kT_0/m)^{1/2} \tag{9}$$

where $T_0 = 2E/3N_0 k$ is the temperature of the plasma when all of E is deposited as heat in a pure Maxwellian. With this choice of v_0 and with g isotropic the normalisation condition (8) ensuring the same number of particles is that $g(\mathbf{u})$ should satisfy

$$\int_0^\infty g(u) u^2 \, \mathrm{d}u = \frac{1}{4\pi} \tag{10}$$

while the constraint ensuring the same total energy becomes

$$\int_0^\infty g(u) u^4 \, \mathrm{d}u = \frac{3}{8\pi}. \tag{11}$$

3 Scaled entropies of specific distribution functions

3.1 Pure Maxwellian $g_M(u)$

Expressed in terms of g, a pure Maxwellian satisfying constraints (10) and (11) is

$$g_M(u) = \frac{e^{-u^2}}{\pi^{3/2}} \tag{12}$$

which, in (7), integrates straightforwardly to give the scaled entropy

$$\Sigma_M = \frac{3}{2}(1 + \ln \pi). \tag{13}$$

3.2 Maxwellian with bump in tail $g_{BIT}(u)$

One of the forms of distribution function with an accelerated component is a Maxwellian with a 'bump in tail'. Here we consider only a simple case to allow analytic treatment, namely a bump in tail centred on speed u_1, and of narrow width $\Delta u \ll 1$ over which the bump contribution to g is a constant added to the Maxwellian component which describes the rest of the plasma (cf Brown (1993)). Because of energy constraint (11), the temperature of this Maxwellian component is reduced by a factor $\tau < 1$ relative to the pure Maxwellian of case 3.1, by the energy resident in the bump. If the fraction of the particles by number in the bump and in the Maxwellian are respectively ϕ and $(1 - \phi)$ (to satisfy the number constraint (10)) then

$$g_{BIT}(u) = \frac{1-\phi}{(\pi\tau)^{3/2}} e^{-u^2/\tau} + \begin{cases} \dfrac{\phi}{4\pi u_1^2 \Delta u} & u \text{ in } \Delta u \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

with

$$\tau = \frac{1 - \frac{2}{3}\phi u_1^2}{1 - \phi} \tag{15}$$

and we note that the condition $u_1^2 \leq 3/2\phi$ has to be met since $\tau \geq 0$, equality holding when the Maxwellian component of $(1 - \phi)N_0$ remains cold.

Substituting (14) and (15) in (7) and approximating the integrals on the assumption that $\Delta u \ll 1$ and $\phi/(4\pi u_1^2 \Delta u) \gg 1 - \phi e^{-u_1^2/\tau}/(\pi\tau)^{3/2}$ (ie, the bump is locally 'large') we obtain for the scaled entropy

$$\begin{aligned} &\Sigma_{BIT}(\phi, u_1, \Delta u) = \\ &\quad -\phi \ln\left(\frac{\phi}{4\pi u_1^2 \Delta u}\right) + \frac{3}{2}(1-\phi) \\ &\quad \times \left[1 + \ln\pi - \frac{5}{3}\ln(1-\phi) + \ln\left(1 - \frac{2}{3}\phi u_1^2\right)\right] \end{aligned} \tag{16}$$

3.3 Pure power law $g_{PL}(u)$

A commonly occurring form of accelerated particle spectrum in solar flares and elsewhere in astrophysics is the power law $v^{-\alpha}$. Such $f(v)$ diverges unless flattened or truncated at small v, so we first consider a truncated pure power law comprising *all* the plasma particles and again with the same total energy as the pure Maxwellian 3.1. With number constraint (10), this has the form

$$g_{PL} = \begin{cases} 0 & u < u_1 \\ \dfrac{\alpha - 3}{4\pi u_1^3}\left(\dfrac{u}{u_1}\right)^{-\alpha} & u \geq u_1 \end{cases} \tag{17}$$

where the dimensionless low energy cut-off speed u_1 has to be related to α to satisfy total energy constraint (11), *viz.*

$$u_1 = \left(\frac{3}{2}\frac{\alpha - 5}{\alpha - 3}\right)^{1/2} \tag{18}$$

and we note that $\alpha > 5$ is required for finite total energy. We note that the relation between α and the spectral index δ normally used (in solar flare physics) to describe the non-relativistic particle flux spectrum differential in kinetic energy (Brown, 1971) is $\alpha = 2\delta + 1$.

Substitution of (17) and (18) in (7) leads to the scaled entropy

$$\Sigma_{PL}(\delta) = \frac{\delta + \frac{1}{2}}{\delta - 1} + \tfrac{1}{2}\ln\left[\frac{27\pi^2}{2}\frac{(\delta-2)^3}{(\delta-1)^5}\right] \tag{19}$$

3.4 Maxwellian with a power law tail

The conventional wisdom regarding the behaviour of the real distribution funtion for the bulk of the electrons in a solar flare is that it is roughly Maxwellian at low velocities (where collisions dominate) but of power law (or other nonthermal form) at high velocities (where mean free paths become long and runaway may occur). Such a particle model spectrum finds some support in recent inversions of high resolution bremsstrahlung hard x-ray spectra (Johns and Lin, 1992; Thompson et al., 1992; Piana, 1994) though these inferences are ambiguous because of the effects of noise (Craig and Brown, 1986) and of averaging over an inhomogeneous source (Brown, 1971). It is therefore instructive to consider such a distribution from the viewpoint of entropy. A convenient functional form which exhibits these asymptotic behaviours at low and high v, and which circumvents most of the analytic messiness associated with a piecewise description is $f(v) \sim (1 + v^2/v_1^2)^{-\alpha/2}$ (Robinson, 1993). The high velocity spectral index is α, as in 3.3, and the temperature T defining the shape at small v is fixed by α and v_1 such that $\frac{1}{2}mv_1^2 = \alpha kT$. Similarly to cases 3.2 and 3.3, α and v_1 have to be interrelated to satisfy constraints (10)

and (11), namely $u_1 = v_1/v_0 = \sqrt{(\alpha - 5)/2}$. Taking these constraints into account, the resulting $g(u)$ is

$$g_{MPL}(u) = \left[\frac{2}{\pi(\alpha-5)}\right]^{3/2} \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha-3}{2}\right)} \left(1 + \frac{2}{\alpha-5}u^2\right)^{-\alpha/2} \tag{20}$$

where $\Gamma(x)$ is the gamma function.

Substitution of (20) in (7) gives, for this case (with $\alpha = 2\delta + 1$ as in Sect. 3.3)

$$\begin{aligned} \Sigma_{MPL}(\delta) \quad = \quad & -\ln\left[\frac{\Gamma(\delta+\frac{1}{2})}{\pi^{3/2}(\delta-2)^{3/2}\Gamma(\delta-1)}\right] \\ & -(\delta+\tfrac{1}{2})[\psi(\delta-1)-\psi(\delta+\tfrac{1}{2})] \end{aligned} \tag{21}$$

where the psi function is

$$\psi(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}[\ln\Gamma(x)]. \tag{22}$$

4 Entropy comparison of nonthermal distributions with a pure Maxwellian

We now compare the entropies of nonthermal distributions (16), (19), and (21) with that of the pure Maxwellian (13).

4.1 Bump-in-tail

This case was already briefly discussed by Brown (1993). Its most important features are:

1. As $\phi \to 0$, the pure Maxwellian entropy (13) is recovered.

2. For *any* ϕ, u_1 (ie, no matter how few particles are accelerated or to how low a speed), the relative entropy $\to -\infty$ as $\Delta u \to 0$ – ie, as a finite number of particles in $f(u)$ is crowded into an arbitrarily narrow velocity range. Considering the relative entropy as the negative of the log of the probability P, this quantifies the qualitative improbability statement in Sect. 1 concerning the relative improbability of nonthermal distributions. We note, however, that the divergence of Σ to $-\infty$ is an artefact of taking the approximate continuum function expression (1) for entropy to a point where the smoothness assumption made breaks down (see Appendix).

3. No matter how small the fraction ϕ of particles accelerated is, as $u_1^2 \to \frac{3}{2}\phi$, $\Sigma_{BIT} \to -\infty$ because the fraction of the total energy in fast particles $\to 1$ at that point, which is arbitrarily improbable. The same remark as in 2 applies.

Figure 1 is in file fig1.ps

Figure 1: Scaled entropy $\Delta\Sigma \equiv \Sigma_M - \Sigma$, relative to the Maxwellian for (a) a pure truncated power law and (b) a Maxwellian with a power-law tail.

4.2 Pure power law

In figure 1a, we show the quantity $\Delta\Sigma_{PL}(\delta) = \Sigma_M - \Sigma_{PL}$ as a function of δ for the pure power-law case. The key features of this curve are

1. There is a minimum in $\Delta\Sigma_{PL}$ which occurs at $\delta = \delta_0 \equiv (3+\sqrt{7})/2 \approx 2.82$. This is the value of δ at which a sharply truncated power law most closely approaches the entropy of the Maxwellian.

2. As $\delta \to \infty$, $\Sigma_{PL} \to -\infty$ and $\Delta\Sigma_{PL} \to +\infty$. This is because as $\delta \to \infty$, more and more of the particles are concentrated at a single velocity $u_1 = \sqrt{3/2}$ (by (18)) and the distribution is more and more like a delta function, with associated low probability (cf., 4.1). The same limitation applies here as in point 2 of Sect. 4.1.

3. As $\delta \to 2$, $\Delta\Sigma_{PL} \to +\infty$ logarithmically. This is because the distribution function then becomes very flat and wide with a decreasing probability of the increasing number of particles needed at $u \to \infty$ to satisfy the total energy constraint (11).

Another way to express the entropy comparison between g_M and g_{PL} is to consider the temperature T_δ (or total energy) which a pure Maxwellian would need to have in order to have

as low an entropy as g_{PL} with the same total number N_0 of particles. The generalisation of (12) to a Maxwellian of temperature $\tau = T/T_0$ is $g_M(u,\tau) = \exp(-u^2/\tau)/(\pi\tau)^{3/2}$ and of (13) is

$$\Sigma_M(\tau) = \tfrac{3}{2}[1 + \ln(\pi\tau)]. \tag{23}$$

Equating (23) with (19) we can find the dimensionless temperature $\tau(\delta) = T_\delta/T_0$ required for the Maxwellian entropy to be as low as the power law, *viz.* $\ln\tau_{PL} = 2[\Sigma_{PL} - \Sigma_M]/3$, or

$$\ln\tau_{PL}(\delta) = \frac{4-\delta}{3(\delta-1)} + \ln\left[\frac{3}{(2\pi)^{1/3}}\frac{\delta-2}{(\delta-1)^{5/3}}\right] \tag{24}$$

which also has a maximum at the value of $\delta = \delta_0$.

The above results allow some quantitative interpretation, albeit somewhat speculative. First, if we consider transient energy release events, then the entropies of distribution functions of different δ should be measures of the relative probabilities of these δs being realised *if* all the energy went into a pure power-law. In a large set of such events these should reflect in some way the relative frequency of occurrence of different δs. In particular, values of δ near δ_0, where Σ_{PL} has a maximum, should be commonest (This is certainly not the case in the observed frequency distribution of flare δ values inferred from hard X-ray bursts, but nor are flare spectra those of pure power laws with no Maxwellian component! – cf. Sect. 4.3).

Second, we note that the above power-law entropy analysis also applies to the relativistic regime when v is replaced by the momemtum p and the maximum entropy power-law spectrum again has index δ_0. This δ_0 is very close to the observed mean spectral index of cosmic ray particles over a very wide energy range (Longair, 1981) – a fact pointed out to us by Bell (1994). That is, if an unspecified mechanism operates to force a power-law form on the distribution function, under conditions of constant total particle number and energy, then the highest entropy (most likely) state is that with a power-law index close to the observed cosmic-ray one. This fact should be explored further.

4.3 Maxwellian with power-law tail

In figure 1b we show the variation with δ of $\Delta\Sigma_{MPL}(\delta) \equiv \Sigma_M - \Sigma_{MPL}$ (equivalent to τ_δ) for this case, defined similarly to those in 4.2, based on equations (13) and (21), *viz.*

$$\begin{aligned}\Delta\Sigma_{MPL}(\delta) \quad &= \quad \frac{3}{2} + \ln\left[\frac{\Gamma(\delta+\frac{1}{2})}{(\delta-2)^{3/2}\Gamma(\delta-1)}\right] \\ &\quad + (\delta+\tfrac{1}{2})[\psi(\delta-1) - \psi(\delta+\tfrac{1}{2})] \quad (25)\\ &\quad (26)\end{aligned}$$

In this case, which is expected to be much closer to the solar flare situation than the pure power law, there is no extremum in $\Delta\Sigma_{MPL}$ which tends to $+\infty$ as $\delta \to 2$ and falls as δ increases. This is essentially because, for a prescribed total N_0 and E, the steep power law tail more and more closely resembles the Maxwellian for large δ. On the other hand as δ decreases, the entropy becomes smaller and smaller as deviation from the Maxwellian increases. If this result is interpreted statistically in terms of the relative likelihood of a tail of given δ occurring, as compared to pure heating, it means that smaller and smaller δ should occur less and less frequently.

Observational inference of δ in solar flares is achieved from the spectral index γ of bremsstrahlung hard X-rays (eg, Brown (1971)). In terms of the electron acceleration spectrum the event integrated γ is related to δ by $\delta = \gamma + 1$ (for collision dominated thick target energy losses) and by $\delta = \gamma - 1$ for collisionally thin targets. Kane (1974) has reported the statistics of observed occurrence of different γ values. The qualitative trend of these data is similar to that predicted from the above entropy argument – ie, a sharp decline of numbers of events of small spectral index and a frequency distribution flattening off at larger indices (though the data are instrumentally limited to indices $\gamma \lesssim 6$).

In view of the simplications made in the present theoretical analysis (homogeneous source volume, neglect of waves and of anisotropy) the trend of the theoretical and observed distributions is remarkably similar, and worthy of further investigation.

5 Conclusions

The lower entropy of a hot plasma with an accelerated component, as compared to a pure Maxwellian of the same total particle number and energy, quantifies the extent to which the hardest non-thermal components are further from equilibrium and so should occur less frequently. Calculation of the entropy difference in the case of a Maxwellian with a power-law tail of index δ leads to a predicted frequency distribution of spectral indices in qualitative agreement with the general trend of the distribution inferred from the statistics of flare hard X-ray bursts.

All distributions except the Maxwellian are non-equilibrium ones, which means that we cannot define a temperature for them and so, in turn, that we cannot use our expression for the entropies of these distributions in any non-trivial purely thermodynamical argument. We can, however, make a cautious statistical argument, as we have done in Sects. 4.2 and 4.3, to the extent that, given a process which produces a certain non-thermal particle distribution, we expect some characteristics of that distribution to be more likely than others. We have no doubt that there are more, and more detailed, arguments to be made using these results.

The close coincidence of the observed cosmic-ray spectral index with that of a maximum entropy pure power-law is likewise tantalising.

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Appendix: the limitations of Eq. (7)

As pointed out in Sect. 4.1, the divergence of the 'scaled entropy' Σ to $-\infty$ is an artefact of the approximations made in deriving (1). To make this clear, we present here a brief derivation of (7) which concentrates on the steps *before* Eq. (1) rather than, as in Sect. 2, on the physics behind the definition of Σ.

As above, we can write (2)

$$f(\mathbf{r},\mathbf{v}) = n(\mathbf{r})G(\mathbf{v}),$$

where $n(\mathbf{r})$ and $G(\mathbf{v})$ are position and velocity distribution functions, respectively. Since these are independent, the statistical weight for the distribution $f(\mathbf{r},\mathbf{v})$ – the number of ways of realising it – is simply $\Omega[f] = \Omega[n(\mathbf{r})] \times \Omega[G(\mathbf{v})]$, so that the entropy of the distribution $f(\mathbf{n},\mathbf{v})$ will be the *sum* of the entropies in position and velocity spaces separately; this means that we can immediately ignore the constant contribution of $n(\mathbf{r})$ to the entropy, and instead concentrate on the entropy of the velocity distribution alone. In these terms, this is

$$S[G] = k \ln \Omega[G], \tag{27}$$

where k is Boltzmann's constant, and $\Omega[G]$ is the statistical weight of the distribution, or the number of ways in which the distribution can be realised. Note that, since $\Omega[G]$ is a positive integer, the entropy $S[G]$ has a minimum of zero (which occurs when $G(\mathbf{v})$ is a delta function – there is only one way of giving all the particles the same velocity).

First define a dimensionless 'velocity' $\mathbf{u} \equiv \mathbf{v}/v_0$, for some arbitrary speed v_0. Imagine dividing the accessible volume of velocity space into a finite number ν of small, but not infinitesimal, volumes, size $\delta^3\mathbf{u}$, at $\mathbf{u}_i$, so that there will be $N_i \equiv G(\mathbf{u}_i)\delta^3\mathbf{u}$ particles in the volume centred on $\mathbf{u}_i$. Consequently, the total number of ways of distributing the N particles amongst the cells will be the multinomial $\Omega = N!/\prod_i N_i!$. If we take $\delta^3\mathbf{u}$ to be large enough that each of the N_i is large, then we can approximate $\ln\Omega$ using Stirling's formula $\ln N! \approx N\ln N - N$, to obtain

$$\ln\Omega \approx -N\sum_i g(\mathbf{u}_i)\ln g(\mathbf{u}_i)\delta^3\mathbf{u}. \tag{28}$$

where we have used the normalisation

$$\sum_i N_i = \sum_i G(\mathbf{u}_i)\delta^3\mathbf{u} = N, \tag{29}$$

and then used the largeness of $G(\mathbf{u}_i)\delta^3\mathbf{u}$ to ignore $\ln\delta^3\mathbf{u}$, compared with $\ln G(\mathbf{u}_i)$, before substituting $G(\mathbf{u}) \equiv Ng(\mathbf{u})$. If, finally, the scale $\delta^3\mathbf{u}$ is small enough that we can take the distribution function $g(\mathbf{u})$ to be constant over it, then we can approximate this final estimate for the distribution function (28) by an integral, and write the entropy (27) as

$$\Sigma[g] \equiv S[g]/Nk = -\int g(\mathbf{u})\ln g(\mathbf{u})\,\mathrm{d}^3\mathbf{u}, \tag{30}$$

where we have replaced the approximation sign by an equality. Compare (7). Crucially, the validity of the approximation (30) depends on the distribution function $G(\mathbf{u})$ being such that there can be in fact a scale $\delta^3\mathbf{u}$ which simultaneously has sufficiently large N_i and sufficiently constant $g(\mathbf{u})$. If this is not so, we generate the artefacts described in Sects. 4.1 and 4.2.

References

Bell, A. R., 1994, personal communication

Brown, J. C., 1971, *Sol. Phys.* **18**, 489

Brown, J. C., 1991, in (Culhane and Jordan, 1991), pp 413–424

Brown, J. C., 1993, in (Uchida et al., 1993)

Brown, J. C. and Smith, D. F., 1980, *Rep. Prog. Phys.* **43**, 125

Craig, I. J. D. and Brown, J. C., 1986, *Inverse Problems in Astronomy*, Adam Hilger (Bristol)

Culhane, J. L. and Jordan, C. (eds.), 1991, *The Physics of Solar Flares*, Vol. 336A of *Phil. Trans. Roy. Soc. London*

Grognard, R. J.-M., 1983, *Phys. Lett.* **95A(8)**, 432

Johns, C. M. and Lin, R. P., 1992, *Sol. Phys.* **137(1)**, 121

Kane, S. R., 1974, IAU Symposium no. 57, pp 105–141, D. Reidel

Longair, M. S., 1981, *High Energy Astrophysics*, Cambridge University Press

Piana, M., 1994, *A&A* **288(3)**, 949

Robinson, P., 1993, personal communication

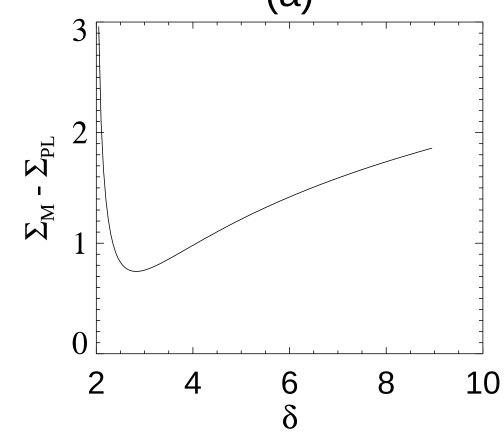
Simnett, G. M., 1991, in (Culhane and Jordan, 1991), pp 439–450

Thompson, A. M., Brown, J. C., Craig, I. J. D., and Fulber, C., 1992, *A&A* **265(1)**, 278

Tolman, R. C., 1959, *Principles of Statistical Mechanics*, International Series of Monographs on Physics, Oxford University Press

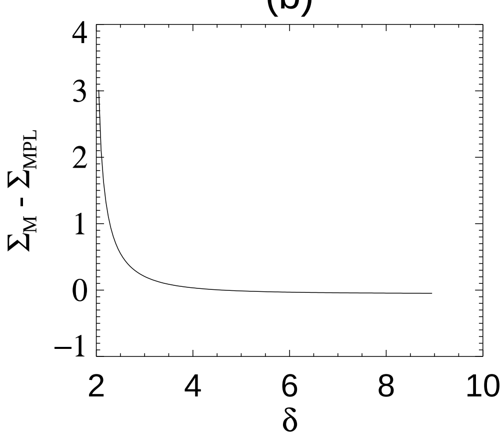
Uchida, Y., Watanabe, T., Shibata, K., and Hudson, H. S. (eds.), 1993, *X-Ray Solar Physics from Yohkoh*, No. 10 in Frontiers Science, Universal Academic Press, Inc

(a)



$\Sigma_M - \Sigma_{PL}$
0
1
2
3
2
4
6
8
10
δ

(b)



$\Sigma_M - \Sigma_{MPL}$
−1
0
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